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Lyapunov Function for the Kuramoto Model of Nonlinearly Coupled Oscillators

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A Lyapunov function for the phase-locked state of the Kuramoto model of nonlinearly coupled oscillators is presented. It is also valid for finite-range interactions and allows the introduction of thermodynamic formalism such as ground states and universality classes. For the Kuramoto model, a minimum of the Lyapunov function corresponds to a ground state of a system with frustration: the interaction between the oscillators, XY spins, is ferromagnetic, whereas the random frequencies induce random fields which try to break the ferromagnetic order, i.e., global phase locking. The ensuing arguments imply asymptotic stability of the phase-locked state (up to degeneracy) and hold for any probability distribution of the frequencies. Special attention is given to discrete distribution functions. We argue that in this case a perfect locking on each of the sublattices which correspond to the frequencies results, but that a partial locking of some but not all sublattices is not to be expected. The order parameter of the phase-locked state is shown to have a strictly positive lower bound ($r \geq 1/2$), so that a continuous transition to a nonlocked state with vanishing order parameter is to be excluded.

KEY WORDS: Nonlinear oscillator; phase locking; Lyapunov function; asymptotic stability; phase transition; collective phenomena; thermodynamic formalism; order parameter.

1. INTRODUCTION

Van der Pol's description of frequency locking[1] has aroused a lasting theoretical interest in all kinds of locking phenomena that occur in nonlinear oscillators. Originally, a single or only a few oscillators were involved; see in particular van der Pol's seminal work on an LC circuit

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with triode and forcing term and its followup. The interest in the *collective* behavior of a large assembly of nonlinear oscillators is more recent. It has been stimulated greatly by a model proposed by Kuramoto,[2–5] who assumed N oscillators coupled "all-to-all" and described by a phase ϕ_i, $1 \leqslant i \leqslant N$, with

$$\dot{\phi}_i = \omega_i - \frac{K}{N} \sum_{j=1}^{N} \sin(\phi_i - \phi_j) \tag{1}$$

Here $\dot{\phi} = d\phi/dt$, $K \geqslant 0$, and the frequencies ω_i are independent, identically distributed random variables. The underlying probability measure on the reals is denoted by μ. In contrast to an extensive part of the existing literature, but in agreement with practical requirements, its support is supposed to be contained in a *bounded* interval.

The Kuramoto model has been studied extensively during recent years. We refer in particular to the beautiful work of Ermentrout and Kopell[6–8] and Strogatz, Mirollo, and Matthews.[9–13] There appears to exist a critical K_c such that for $K > K_c$ the system is in a *phase-locked* state characterized by $\dot{\phi}_i = \dot{\phi}$ for all $i = 1,..., N$, whereas no such state exists for $K < K_c$. Instead, one then encounters a partially coherent state or, as we will see below, a state which is not coherent at all.

Though the model (1) looks quite simple, appearances are deceiving. Here we concentrate on phase locking and exhibit a Lyapunov function $\mathscr{H}$. The phase-locked state has long been known,[2–13] but until now stability proofs, if any, have been hard to obtain. We show that a phase-locked state is a minimum of $\mathscr{H}$ and thus (quite) unique and (asymptotically) stable. Furthermore, we are able to offer a physical interpretation of this state and its stability.

Since the existence of a Lyapunov function greatly simplifies the mathematics of a stability analysis, we have made some effort to keep the paper self-contained. Moreover, we focus on *discrete* distributions of the ω_i. They have hardly been considered so far and give rise to interesting physics. We would like to stress, however, that the validity of our analysis does not depend in any way whatsoever on the probability distribution being discrete. This is due to the underlying mathematical mechanism: the strong law of large numbers,[14] which is valid for *any* probability distribution.

In the following sections we introduce our Lyapunov function $\mathscr{H}$, determine the phase-locked state as its minimum, and study the associated fixed-point equations. The Lyapunov function is not specific to the Kuramoto model, which is of mean-field or infinite-range type. It holds for *any* model with symmetric interactions of finite instead of infinite range. For the Kuramoto model, the Lyapunov function $\mathscr{H}$ represents the energy of an XY ferromagnet of strength K in a random field induced by the random distribution of the frequencies ω_i. As such, the system is *frustrated* in that the XY ferromagnet likes to have all spins parallel, i.e., all ϕ_i equal and thus perfectly phase locked, whereas the random field $(\omega_i - \langle\omega\rangle)$ tries to break the ferromagnetic order. As K decreases, the model exhibits a phase transition at K_c: For $K > K_c$ the ferromagnet wins and the system is totally phase locked, whereas for $K < K_c$ the random field takes over. We estimate both K_c and the range of the order parameter r that describes the macroscopic extent of the phase locking. We study several examples, face the question of what happens if we are given a *discrete* frequency distribution and some *but not all* of the oscillators can lock, and finally discuss the salient differences between the present, more general type of model and the "generic" one with absolutely continuous, symmetric distributions such as the Gaussian and the Lorentzian.[2–5,7–13] It will turn out that the latter type of model does not behave in a truly generic way, since the *partial* locking,[3,12] which shows up for K in an open interval just below K_c, is absent in models with a discrete distribution. In other words, different universality classes exist. We also touch upon the relevance of locking in the present model to the question of what coherent oscillations in the cortex are good for, a hotly debated topic in theoretical neurobiology.

2. LYAPUNOV FUNCTION

The dynamics (1) allows an interesting *sum rule.* We add the $\dot{\phi}_i$, divide by N, and find

$$\frac{d}{dt}\left(N^{-1}\sum_{i=1}^{N}\phi_i\right)=\left(N^{-1}\sum_{i=1}^{N}\omega_i\right)-\frac{K}{N^2}\sum_{i,j}\sin(\phi_i-\phi_j) \tag{2}$$

Since the sine is an odd function, the second sum on the right vanishes (interchange i and j) and we are left with

$$\frac{d}{dt}\left(N^{-1}\sum_{i=1}^{N}\phi_i\right)=N^{-1}\sum_{i=1}^{N}\omega_i\equiv\langle\omega\rangle \tag{3}$$

As $N\to\infty$ the quantity $\langle\omega\rangle$ is a nonrandom number and equals the mean $\int d\mu(\omega)\omega$ by the strong law of large numbers.[14] If, then, we have phase locking defined by $\dot{\phi}_i=\dot{\phi}$, $1\leqslant i\leqslant N$, we are bound to find $\dot{\phi}=\langle\omega\rangle$.

We now introduce new variables φ_i defined by

$$\varphi_i=\phi_i-\Omega t \tag{4}$$

where Ω is at our disposal.

In terms of the φ_i the equations of motion (1) reappear in the form

$$\dot{\varphi}_i = (\omega_i - \Omega) - \frac{K}{N} \sum_j \sin(\varphi_i - \varphi_j) \tag{5}$$

Suppose for a moment that there were no randomness, so that $\omega_i = \omega$ for $1 \leqslant i \leqslant N$. If we choose $\Omega = \omega$, then we arrive at a simple gradient dynamics,[5]

$$\dot{\varphi}_i = -\frac{K}{N} \sum_j \sin(\varphi_i - \varphi_j) = -\frac{\partial \mathscr{H}}{\partial \varphi_i} \Rightarrow \dot{\boldsymbol{\varphi}} = -\nabla \mathscr{H} \tag{6}$$

with

$$\mathscr{H} = -\frac{K}{2N} \sum_{i,j} \cos(\varphi_i - \varphi_j) = -\frac{K}{2N} \sum_{i,j} \mathbf{S}_i \cdot \mathbf{S}_j \tag{7}$$

$\mathscr{H}$ is the Hamiltonian of an *XY* ferromagnet. The spins $\mathbf{S}_i$ are unit vectors and the dynamics (6) is a gradient dynamics with $\mathscr{H}$ as Lyapunov function, viz.

$$\dot{\mathscr{H}} = \sum_i \frac{\partial \mathscr{H}}{\partial \varphi_i} \dot{\varphi}_i = -\sum_i \left(\frac{\partial \mathscr{H}}{\partial \varphi_i} \right)^2 = -\|\nabla \mathscr{H}\|^2 \leqslant 0 \tag{8}$$

The inequality in (8) is strict, unless we reach a minimum of $\mathscr{H}$, where $\nabla \mathscr{H} = 0$. Since $\mathbf{S}_i \cdot \mathbf{S}_j \leqslant 1$, a minimum of (7) is reached as soon as $\mathbf{S}_i \cdot \mathbf{S}_j = 1$ for all i and j, i.e., when all spins are *parallel*. Therefore, asymptotically $\varphi_i(t) \to \varphi_\infty$ for all i and we obtain a *perfect* phase locking. In terms of the original variables we have $\phi_i(t) = \omega t + \varphi_\infty$, $1 \leqslant i \leqslant N$.

Is the minimum for $\mathscr{H}$ unique? No, not quite. Due to (7) we can write

$$\mathscr{H} = -\tfrac{1}{2} KN \left(N^{-1} \sum_{i=1}^{N} \mathbf{S}_i \right)^2 \tag{9}$$

which is evidently invariant under a *uniform* rotation of all the $\mathbf{S}_i$ or, in terms of the φ_is, under the transformation $\varphi_i \to \varphi_i + \alpha$, $1 \leqslant i \leqslant N$. As will be shown in Section 5, all eigenvalues of the stability matrix are strictly negative, except for one, which vanishes. In this way the rotational invariance of (9) is taken care of. A minimum is stable and, orthogonally to this direction, asymptotically stable.[15] We now turn to the case of a *non*degenerate distribution of the ω_i.

Also for (5) a Lyapunov function exists. We can, and will, define $\varphi_i \bmod 2\pi$. Equation (5) tells us quite explicitly that there is no harm in doing so. Then

$$\mathscr{H} = -\frac{K}{2N} \sum_{i,j} \cos(\varphi_i - \varphi_j) - \sum_i (\omega_i - \Omega)\, \varphi_i \tag{10}$$

induces a gradient dynamics for (5). Hence $\mathscr{H}$ is a Lyapunov function and the dynamics (5) converges to a minimum of $\mathscr{H}$, if it exists. One might object that restricting $\varphi \bmod 2\pi$ to $[-\pi, \pi]$ is an artefact. If we start far away from a minimum, some of the φ_i may hit the border and jump from $-\pi$ to π or conversely. That does change the second term on the right in (10). No jumping occurs, however, if a minimum of $\mathscr{H}$ can be localized *in the interior* of $[-\pi, \pi]^N$. In a suitable neighborhood, the system then converges to the minimum and we even have asymptotic stability. It is also plain that the idea which has led us to (10) is equally valid if the mean-field interaction $K/2N$ is replaced by a finite-range interaction J_{ij}. Since the modifications of the arguments below are straightforward, they will not be spelled out here.

An extremum of $\mathscr{H}$ is characterized by $\nabla\mathscr{H} = 0$, i.e., by the fixed-point equation

$$0 = (\omega_i - \Omega) - \frac{K}{N} \sum_j \sin(\varphi_i - \varphi_j) \tag{11}$$

for $1 \leqslant i \leqslant N$. Summing over i, we obtain

$$\Omega = N^{-1} \sum_{i=1}^{N} \omega_i = \langle \omega \rangle \tag{12}$$

This determines Ω and is consistent with the observation following the sum rule (3). For a finite-range interaction, exactly the same argument holds, including the sum rule, if the J_{ij} are symmetric, i.e., $J_{ij} = J_{ji}$. We now continue with the Kuramoto model.

Let us denote the difference between ω_i and $\Omega = \langle \omega \rangle$ by $\Delta(\omega) = \omega_i - \langle \omega \rangle$. To solve (11), viz.

$$\Delta(\omega_i) = \frac{K}{N} \sum_j \sin(\varphi_i - \varphi_j) \tag{13}$$

we introduce an order parameter r and an associated variable ψ through[2–4]

$$re^{i\psi} = N^{-1} \sum_{j=1}^{N} e^{i\varphi_j} \tag{14}$$

Since the right-hand side is a convex combination of complex numbers in the convex unit disk, $r \exp(i\psi)$ is in the unit disk itself and $0 \leqslant r \leqslant 1$. Using (14) and $\sin(x) = [\exp(ix) - \exp(-ix)]/2i$, we rewrite (6)

$$\begin{aligned} \dot{\varphi}_i &= \Delta(\omega_i) - \frac{K}{2Ni} \sum_{j=1}^{N} [e^{i(\varphi_i - \varphi_j)} - e^{-i(\varphi_i - \varphi_j)}] \\ &= \Delta(\omega_i) - \frac{K}{2i} [e^{i(\varphi_i - \psi)} - e^{-i(\varphi_i - \psi)}] \end{aligned}$$

so that

$$\dot{\varphi}_i = \Delta(\omega_i) - Kr\sin(\varphi_i - \psi) \tag{15}$$

This equation explicitly tells us that $\dot{\varphi}_i$ is governed by both $\Delta(\omega_i)$ and the *collective* variables r and ψ. The fixed-point equation (11) now assumes the simple form

$$\Delta(\omega_i) = Kr\sin(\varphi_i - \psi) \tag{16}$$

It is basic to all that follows.

For the sake of simplicity we suppose that the ω_i assume only finitely many values $\{\omega\}$ with probabilities $\{p(\omega)\}$. We then can introduce[16] *sublatttices* $I(\omega) = \{i; \omega_i = \omega\}$ consisting of all i with $\omega_i = \omega$. By the strong low of large numbers[14] the size $|I(\omega)|$ of the sublattice $I(\omega)$ is given by $|I(\omega)| \sim p(\omega)N$ as $N \to \infty$.

On a particular sublattice $I(\omega)$ all $\Delta(\omega_i)$ have the same value $\Delta(\omega)$. According to (16), all φ_i also assume the very same value $\varphi(\omega)$ given by

$$\varphi(\omega) - \psi = \arcsin(\Delta(\omega)/Kr) \tag{17}$$

We will verify shortly that the order parameter r is such that $|\Delta(\omega)/Kr| \leqslant 1$ and that arcsin(0) should be 0 *and not* π. By (14) and (17) we have

$$re^{i\psi} = \sum_{\{\omega\}} p(\omega)\exp\left\{i\left[\psi + \arcsin\left(\frac{\Delta(\omega)}{Kr}\right)\right]\right\} \tag{18}$$

and, since $r \geqslant 0$,

$$r = \sum_{\{\omega\}} p(\omega)\cos\left[\arcsin\left(\frac{\Delta(\omega)}{Kr}\right)\right] \tag{19}$$

If $\Delta(\omega) \equiv 0$, then the system is in a perfectly locked state ($\psi = \varphi_\infty$) associated with an energy minimum of the XY ferromagnet and $r = \sum_{\{\omega\}} p(\omega) = 1$ is a *stable* solution. That is, arcsin(0) has to vanish, and

$$\cos\left[\arcsin\left(\frac{\Delta(\omega)}{Kr}\right)\right] = \left[1 - \left(\frac{\Delta(\omega)}{Kr}\right)^2\right]^{1/2} \tag{20}$$

Combining this with (19), we arrive at the fixed-point equation

$$r = \sum_{\{\omega\}} p(\omega)\left[1 - \left(\frac{\Delta(\omega)}{Kr}\right)^2\right]^{1/2} \tag{21}$$

By construction, a solution r is bound to be such that $|\Delta(\omega)/Kr| \leqslant 1$. For $\Delta(\omega) \equiv 0$ (or $K = \infty$) Eq. (21) has the solution $r = 1$. For $|\Delta(\omega)|$ small (or K large but finite) we then also obtain a solution by the implicit function theorem.

For given $\Delta(\omega)$ one may wonder how small K can be chosen ($K \geqslant K_c$) and what is the nature of the transition at K_c where r ceases to exist as a solution of (21). Putting $x = (Kr)^2 \in [0, K^2]$, we can rewrite (21) in the form

$$K^{-1}x = \sum_{\{\omega\}} p(\omega)[x - \Delta^2(\omega)]^{1/2} \equiv \vartheta(x) \tag{22}$$

where $\vartheta(x)$ is defined for $x \geqslant \Delta_m^2 = \sup_\omega \Delta^2(\omega)$. On its domain, ϑ is a convex combination of concave functions and thus *concave* itself. Moreover, Eq. (22) tells us that, as we decrease K, there exists a critical K_c such that we find a (stable) solution for $K > K_c$ and no solution for $K < K_c$. Hence there exists *no* global phase locking for $K < K_c$. Some examples can be found in Section 4. The critical K_c itself can be obtained from (22),

$$K_c^{-1} = \sup_{x > \Delta_m^2} \sum_{\{\omega\}} p(\omega)[x - \Delta^2(\omega)]^{1/2} x^{-1} \tag{23}$$

The restriction $x > \Delta_m^2$ is irrelevant for a discrete distribution, since in that case the function ϑ is strictly concave and such that the maximum in (23) is assumed for $x > \Delta_m^2$. We refer to Ermentrout[6] for an elegant discussion of a simpler case, viz., a continuous symmetric distribution.

Stepping back for an overview, we now want to interpret the Lyapunov function (10) as a Hamiltonian so that a minimum of $\mathscr{H}$ represents a ground state of a physical spin system. The first term on the right is the energy of an XY ferromagnet of mean-field type with coupling strength K. This term aims at keeping *all* spins parallel. The second term represents a kind of *random field* with strength $(\omega_i - \langle\omega\rangle)$ and mean $\langle\omega - \langle\omega\rangle\rangle = 0$. To minimize the energy, the second term wants to make the φ_i with $\omega_i - \langle\omega\rangle > 0$ positive and those with $\omega_i - \langle\omega\rangle < 0$ negative—as positive and negative as possible. So the two terms counteract each other. There is *frustration.* For K large the ferromagnet wins in that the system is phase locked, the sublattices are homogeneous, and, as K decreases, their phases $\varphi(\omega)$ rotate away slowly from a single fixed direction, a minimal energy configuration of the XY ferromagnet. This is brought out clearly by (17). The minimum of $\mathscr{H}$ performs a kind of "unfolding" in the phase space $[-\pi, \pi]^N$—like a multiwinged butterfly, the phases being the wings. Herewith the stability of the phase-locked state obtains a natural explanation. As K decreases further and reaches K_c, the

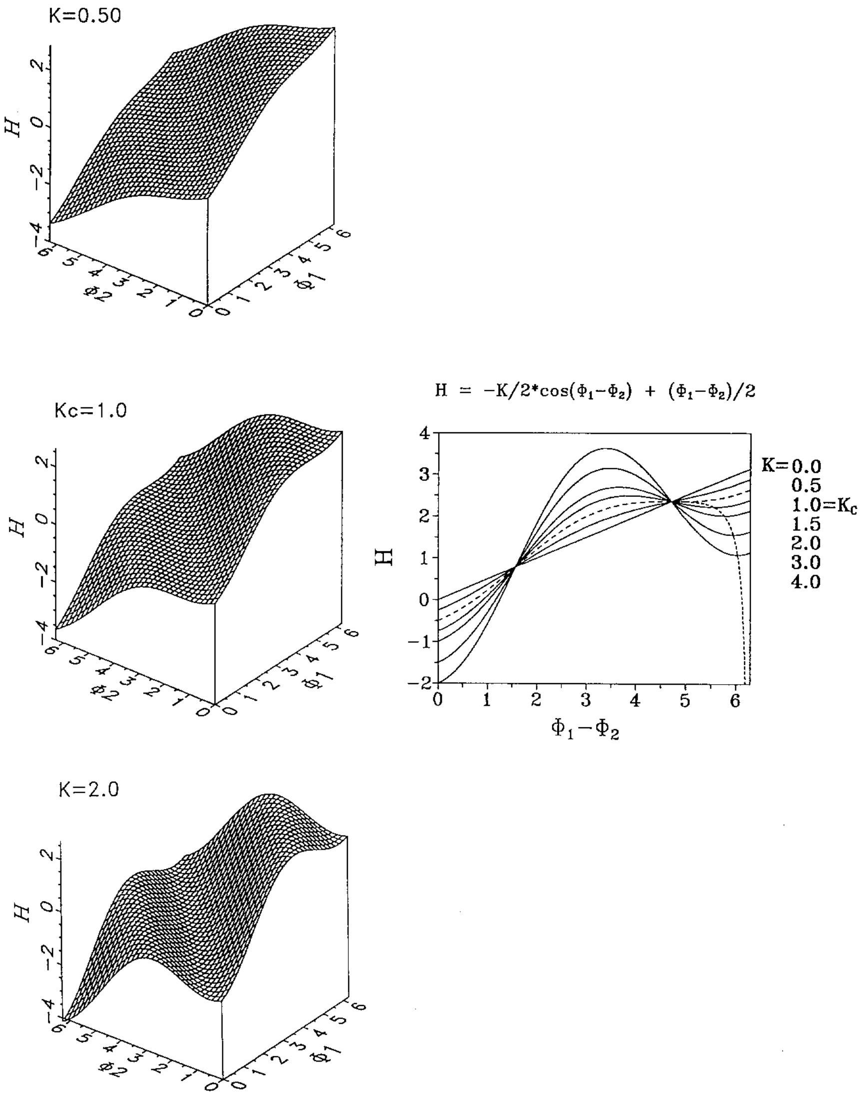

Fig. 1. Left column. For $N=2$, $\mathscr{H}$ has been plotted as a function of φ_1 and φ_2 on $[0, 2\pi] \times [0, 2\pi]$ for various values of K. The frequencies are $\omega_1 - \Omega = -1/2$ and $\omega_2 - \Omega = 1/2$. The motion of the system is a gradient dynamics which evolves in a plane perpendicular to the line $\varphi_1 = \varphi_2$. The intersection with $\mathscr{H}$ contains the trajectory. For $K > K_c$, two ripples occur in the surface and the system gets phase locked in the minimum of one of them. An intersection with the upper ripple (right-hand corner, with $0 \leqslant \varphi_1 - \varphi_2 \leqslant 2\pi$) has been plotted in the right column. The dashed line on the right going downward indicates the location of the minima as K varies.

random field takes over. No solution to (16) and no global phase locking exist beyond K_c anymore.

It may be advantageous to picture the transition at K_c. To this end we take $N=2$ in (10), with $\omega_1 - \Omega = -1/2$ and $\omega_2 - \Omega = 1/2$, and use the third dimension to plot $\mathscr{H}$; see Fig. 1. $\mathscr{H}$ is a function of $\varphi_1 - \varphi_2$ and thus is rotationally invariant. The dynamically relevant direction is orthogonal to $\varphi_1 - \varphi_2 = \text{const}$. Since (8) and (10) induce a gradient dynamics, only the latter direction is relevant. For $K > K_c = 1$, the Lyapunov function has ripples and the system always gets stuck in a minimum of

$$\mathscr{H} = -\frac{K}{2}\cos(\varphi_1 - \varphi_2) + \frac{1}{2}(\varphi_1 - \varphi_2)$$

For $K < K_c$, there is no ripple and, hence, no locking. In the following three sections we will estimate K_c and the order parameter r, consider some examples, and study the stability of the phase-locked state in more detail.

3. ESTIMATING CRITICALITY

One of the main problems is estimating r_c and K_c, the critical values of r and K, in the limit $N \to \infty$. For x large, we can estimate the right-hand side of (23) by modifying an argument of Ermentrout's.[6] We do this for a general probability measure μ and write $K_c = \sup_x \Theta(x)$, where $x \geqslant \Delta_m^2$ and

$$\Theta(x) = x^{-1} \int d\mu(\omega)[x - \Delta^2(\omega)]^{1/2} = x^{-1}\vartheta(x) \tag{24}$$

Computing the derivative Θ', we find

$$\Theta'(x) = \frac{1}{2x^2} \int d\mu(\omega) \frac{2\Delta^2(\omega) - x}{[x - \Delta^2(\omega)]^{1/2}} \tag{25}$$

Hence $\Theta'(x) < 0$ and $\Theta(x)$ is *de*creasing for $x = (Kr)^2$ beyond $2\Delta_m^2$. Thus we obtain the estimate

$$\Delta_m \leqslant (Kr)_c \leqslant \sqrt{2}\,\Delta_m \tag{26}$$

We now turn to lower bounds for K_c and r separately.

Since the square root is a concave function, we apply Jensen's inequality[17] to (22) and find

$$K^{-1}x \leqslant \left\{\int d\mu(\omega)\,[x - \Delta^2(\omega)]\right\}^{1/2} = [x - \langle \Delta^2(\omega) \rangle]^{1/2} \tag{27}$$

and after squaring this,

$$K^{-2}x^2 - x + \langle \Delta^2(\omega) \rangle \leqslant 0 \tag{28}$$

The condition (28) can be realized only if the discriminant is positive, i.e., $K^2 \geqslant 4\langle \Delta^2(\omega) \rangle$. Thus we find

$$K_c \geqslant 2\langle \Delta^2(\omega) \rangle^{1/2} \tag{29}$$

We cannot do better, since the inequality (29) becomes an equality in case $p(\omega_1) = p(\omega_2) = 1/2$, as we will see shortly.

To derive a lower bound for r we start with (26), viz., $\Delta_m \leqslant (Kr)_c$. Though this inequality does not look optimal, it actually is. In the next section we will see that it is saturated by the uniform distribution. If so, we now need a lower bound for K_c^{-1}. To this end we combine (23) and (26), restrict x to the interval $[\Delta_m^2, 2\Delta_m^2]$, and evaluate the right-hand side of (23) at $x = 2\Delta_m^2$ so as to get

$$\begin{aligned} K_c^{-1} &\geqslant \sum_{\{\omega\}} p(\omega)[2\Delta_m^2 - \Delta^2(\omega)]^{1/2}/2\Delta_m^2 \\ &= \sum_{\{\omega\}} p(\omega)\left[2 - \left(\frac{\Delta(\omega)}{\Delta_m}\right)^2\right]^{1/2} \Big/ 2\Delta_m \geqslant (2\Delta_m)^{-1} \end{aligned} \tag{30}$$

Thus we arrive at the extremely simple inequality

$$r_c \geqslant \frac{\Delta_m}{K_c} \geqslant \frac{1}{2} \tag{31}$$

It tells us explicitly that a continuous transition from the phase-locked to a nonlocked state with vanishing r is to be excluded. Note that in obtaining (31) we have not made any special assumption concerning the probability distribution of the frequencies ω_i. Neither do we assert that r must vanish for $K < K_c$. There is just no global phase locking.

The inequality (31) also provides us with an upper bound for K_c in that $K_c \leqslant 2\Delta_m$. In case $p(\omega_1) = p(\omega_2) = 1/2$, this upper bound and the lower bound (29) coincide, so that the upper bound is optimal as well.

4. EXAMPLES

The simplest nontrivial distribution is the one with two frequencies ω_1 and $\omega_2 > \omega_1$ and probabilities $p(\omega_2) = p$ and $p(\omega_1) = 1 - p$. Then $\Delta(\omega_1) = -(\omega_2 - \omega_1)p \leqslant 0$ and $\Delta(\omega_2) = (\omega_2 - \omega_1)(1 - p) \geqslant 0$, while

$$\langle \Delta^2(\omega) \rangle = p(1 - p)(\omega_2 - \omega_1)^2 \tag{32}$$

For $p=1/2$ we get

$$\langle \Delta^2(\omega)\rangle^{1/2}=\tfrac{1}{2}(\omega_2-\omega_1)=|\Delta(\omega)|=\Delta_m \tag{33}$$

The fixed-point equation (22) takes the form

$$K^{-1}x=(1-p)\{x-[p(\omega_2-\omega_1)]^2\}^{1/2}+p\{x-[(1-p)(\omega_2-\omega_1)]^2\}^{1/2} \tag{34}$$

with $x=(Kr)^2\geqslant \Delta_m^2$ and $\Delta_m=\max\{(\omega_2-\omega_1)p,\ (\omega_2-\omega_1)(1-p)\}$. Whatever p, there is a remarkably simple expression for the phase difference,

$$\varphi(\omega_2)-\varphi(\omega_1)=\arcsin[(\omega_2-\omega_1)/K] \tag{35}$$

which can be proven, e.g., by combining an addition formula for two arcsines with (21).

We now return to the case $p=1/2$. Then (34) can be squared so as to give

$$K^{-2}x^2-x+[\tfrac{1}{2}(\omega_2-\omega_1)]^2=0 \tag{36}$$

We get a positive solution to (36) as long as its discriminant is positive, i.e.,

$$K\geqslant K_c=|\omega_2-\omega_1| \tag{37}$$

At K_c we find $x(K_c)=\frac{1}{2}K_c^2$ so that $r(K_c)=\frac{1}{2}\sqrt{2}$. Taking into account both (37) and (33), one easily verifies that inequality (29) has been turned into an equality; in short, it is optimal. Combining (35) and (37), we get that in this particular case and at K_c

$$\varphi_{\max}-\varphi_{\min}=\arcsin(1)=\frac{\pi}{2} \tag{38}$$

The limit $p\to 0$ can also be handled analytically. Here $\Delta(\omega_2)\gg|\Delta(\omega_1)|$ and (34) may be approximated by

$$K^{-1}x=(1-p)\sqrt{x}+p[x-(\omega_2-\omega_1)^2]^{1/2} \tag{39}$$

with $x\geqslant(\omega_2-\omega_1)^2$. Figure 2 shows that in this limit, $K_c=|\omega_2-\omega_1|$ once more, so that $r_c=1$, as is to be expected. We explicitly see that the side condition $x\geqslant\Delta_m^2$ is harmless. Equation (35) implies that here, too, (38) holds. Note that the limit $p\to 0$ is different from the case $p=0$. The latter has perfect locking *whatever* K. Furthermore, the above results for $p=0$, 1/2, and 1 [due to the symmetry $p\to(1-p)$] suggest that (37) and (38) hold for any p. That is indeed the case.

Proposition. For the bimodal distribution with $\omega_2 > \omega_1$ and $p(\omega_2) = p$ we have $K_c = \omega_2 - \omega_1$, whatever p. Moreover, at K_c, the phases of the two sublattices belonging to ω_1 and ω_2 are orthogonal, i.e., (38) holds.

Proof. Turning to (22), we note that in the present case $\vartheta(x)$ starts with a square-root singularity, is monotonically increasing and strictly concave for $x \geqslant \Delta_m^2$, that the side condition is irrelevant when we apply (23), and that the maximum is unique—as is exemplified by Fig. 2. Putting the derivative of the right-hand side of (23) equal to zero, we state as a *fait accompli* that the unique $x(p)$ maximizing (23) equals

$$x(p) = [p^2 + (1-p)^2](\omega_2 - \omega_1)^2$$

A little algebra then suffices to verify $K_c = \omega_2 - \omega_1$ and, taking advantage of (35), we find (38). ▮

Another interesting soluble case has $\omega_1 < \omega_2 < \omega_3$ and $(\omega_2 - \omega_1) = (\omega_3 - \omega_2) = \Delta$ while $p(\omega_1) = p(\omega_3) = p \leqslant 1/2$. In particular, as $\Delta \to \infty$, one finds $K_c \sim p^{-1}\Delta$ and $r_c = p\sqrt{2}$ so that, at K_c, the relative phases are $-\pi/4$, 0, and (38) holds again. This suggests that (38) might be generally true. Our final example shows that this is not the case.

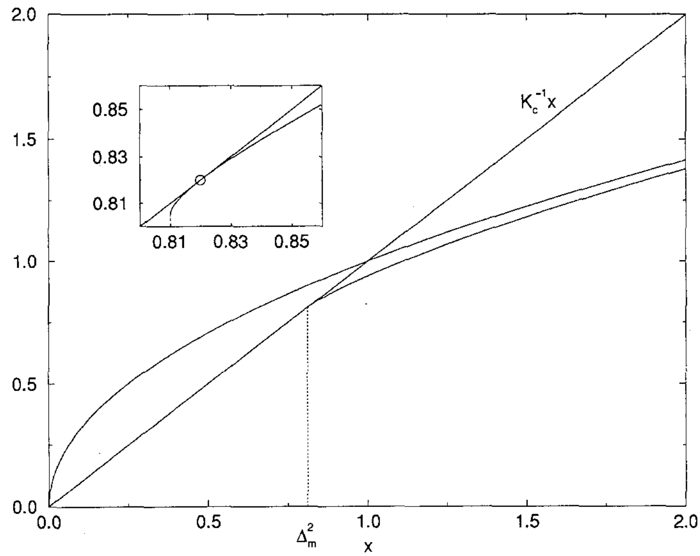

Fig. 2. Graphical solution to the fixed-point equation (34) with $\omega_2 - \omega_1 = 1$. The lower curve and the vertical dashed line represent the case $p = 0.1$. The inset shows why here the side condition $x \geqslant \Delta_m^2$ is irrelevant. Note that the limit $p \to 0$, i.e., the square root with $x \geqslant (\omega_2 - \omega_1)^2$, differs from the case $p = 0$, viz., the square root with $x \geqslant 0$.

The uniform distribution on $[-1, 1]$ is a favorite of the literature. Its fixed-point equation is $(x \geqslant \Delta_m = 1)$

$$K^{-1}x = \tfrac{1}{2}\int_{-1}^{1} d\omega \, (x - \omega^2)^{1/2} \tag{40}$$

The integral can be done exactly and

$$K_c^{-1} = \sup_{x \geqslant 1} \tfrac{1}{2}[x^{-1}(x-1)^{1/2} + \arcsin(1/x^{1/2})] \tag{41}$$

One either applies an argument of Ermentrout's[6] to (40) or checks explicitly that the right-hand side of (41) assumes its maximum at $x = (Kr)^2 = 1$, so that $K_c = 4/\pi$. In addition, Eq. (17) implies $\varphi_{\max} - \varphi_{\min} = \pi$.

5. STABILITY

Before embarking on a more detailed stability analysis it is nice to see what can be said beforehand. To this end, we return to the XY ferromagnet (7). Due to the gradient dynamics (6), the system relaxes to a minimum of $\mathscr{H}$ which is characterized by $\varphi_i = \varphi_\infty$ for $1 \leqslant i \leqslant N$. Is it unique? No, as we have seen, it is not. The ground state of (7) is rotationally invariant and it remains so if we add the random field to $\mathscr{H}$ so as to arrive at (10). The reason is that a uniform rotation through α produces an extra term

$$\frac{\alpha}{N}\sum_{i=1}^{N} (\omega_i - \langle \omega \rangle) = 0$$

which vanishes by the very definition (3) of $\langle \omega \rangle$. Thus we expect a permanent eigenvalue zero belonging to the eigenvector (1, 1,..., 1) of the Jacobian matrix $\mathbb{D}$ at a fixed point of the equations of motion

$$\dot{\varphi}_i = \Delta(\omega_i) - \frac{K}{N}\sum_{j=1}^{N} \sin(\varphi_i - \varphi_j) \tag{42}$$

Doing the calculation explicitly and noting that there is no harm in dropping the prefactor K/N, we find $\cos(\varphi_i - \varphi_j)$ for $i \neq j$ and $-\sum_{j(\neq i)} \cos(\varphi_i - \varphi_j)$ for $i = j$. For the XY ferromagnet without random field, all $(\varphi_i - \varphi_j)$ vanish and $\mathbb{D} = -N\mathbb{1} + \mathbf{1}$, where $\mathbb{1}$ is the unit matrix and $\mathbf{1}$ is the matrix with all elements equal to 1. The matrix $\mathbf{1}$ has a single, nondegenerate eigenvalue N with eigenvector (1, 1,..., 1) corresponding to a uniform rotation of the original system, and $(N-1)$ eigenvalues 0. Thus $\mathbb{D}$ has a single eigenvalue 0, as predicted, and all other eigenvalues equal $-N$. In passing we note that, whatever the fixed point, (1, 1,..., 1) is an eigenvector of $\mathbb{D}$ with eigenvalue zero.

As the $|\Delta(\omega_i)|$ increase, the phases $\varphi(\omega_i)-\psi$ "unfold" continuously; cf. (16). Since the matrix elements of $\mathbb{D}$ are continuous functions of the $\Delta(\omega_i)$, so are its eigenvalues[18] and, consequently, the fixed point originating from $r=1$ at $K=\infty$ remains stable for quite a while. In fact, as long as the $\cos(\varphi_i-\varphi_j)$ remain positive, i.e., $\varphi_{\max}-\varphi_{\min}\leqslant\pi/2$, we can apply the Gershgorin disk theorem[19] so as to conclude that the eigenvalues of $\mathbb{D}$ are negative. Figuring out what the spectrum of $\mathbb{D}$ exactly looks like is quite hairy. To prove that the fixed point originating from $r=1$ remains stable down to K_c, it is simpler, and also more physical, to note that a loss of stability means that an eigenvalue of $\mathbb{D}$ moves through zero at a positive rate. Hence[20] a new fixed point has to bifurcate from the old one. Since the function ϑ in (22) is concave and increasing, this does not happen. So we are done.

It may be clarifying to consider a simple example, viz., the case $p(\omega_1)=p(\omega_2)=1/2$. The fixed-point equation (36) has two roots which for small $\omega_2-\omega_1$ (or large K) lead to two values for the order parameter r,

$$r_+=1-\frac{1}{2}\left(\frac{\omega_2-\omega_1}{K}\right)^2, \qquad r_-=\frac{\omega_2-\omega_1}{2K} \tag{43}$$

The phases of the states corresponding to r_+ and r_- have been indicated in Fig. 3. They allow a simple interpretation. In the limit $(\omega_2-\omega_1)/K\to 0$, the first corresponds to *all* spins parallel. It is a stable ground state. The second has the spins on both $I(\omega_1)$ and $I(\omega_2)$ parallel, but $\varphi(\omega_2)-\varphi(\omega_1)\approx\pi$, i.e., the sublattices have their spins *antiparallel* and the total magnetization vanishes. This configuration corresponds to an energy *maximum* of the XY ferromagnet. It is a stationary point ($\nabla\mathscr{H}=0$) but evidently an *un*stable one; cf. (9). As $(\omega_2-\omega_1)/K$ increases, $\varphi(\omega_2)-\varphi(\omega_1)$ increases for the stable configuration and decreases for the unstable one. The phases of the stable and the unstable configurations meet each other—at least in this case—at $\varphi=\pi/4$ and $\varphi=-\pi/4$, respectively. That is, they meet at K_c. As they merge, the phase-locked state disappears. We now reinterpret this in terms of the Jacobian $\mathbb{D}$ at a stable fixed point.

We take N even. As $N\to\infty$, we have at a fixed point

$$\mathbb{D}=-\frac{N}{2}(1+\varepsilon)+\begin{pmatrix}1 & \varepsilon\\ \varepsilon & 1\end{pmatrix}\otimes\mathbf{1} \tag{44}$$

where $\varepsilon=\cos[\varphi(\omega_2)-\varphi(\omega_1)]$ and $\mathbf{1}$ is an $N/2\times N/2$ matrix with all elements equal to one. Furthermore, $\otimes$ denotes a direct or Kronecker product.[21] If σ represents the set of eigenvalues (the spectrum), we

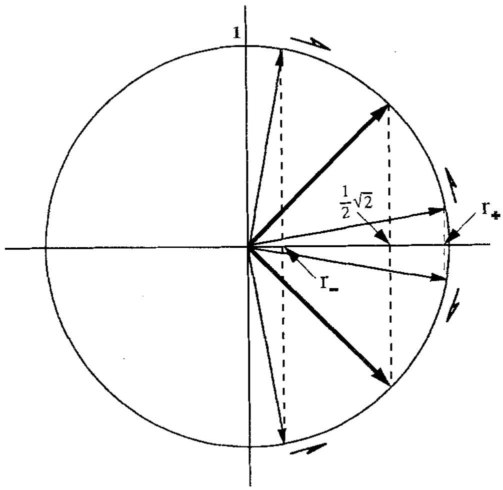

Fig. 3. For large K, a system with $p(\omega_1) = \omega_2 = 1/2$ gives rise to two stationary points of the Lyapunov function $\mathscr{H}$ with order parameter values r_+ and r_-. The former corresponds to a minimum of $\mathscr{H}$ and is stable, whereas the latter corresponds to a maximum of $\mathscr{H}$ and is unstable. The phases are given by the angles with the positive horizontal axis and $r_\pm$ is a convex combination of the r values of the two sublattices with weight 1/2. So $r_+ \approx 1$ and $r_- \approx 0$ and the two sublattices have their block spins nearly parallel or antiparallel. As K decreases, the two upper and lower phases approach each other and they meet at K_c at angles $\varphi_{max} = \pi/4$ and $\varphi_{min} = -\pi/4$. In other words, at K_c they merge, $\varphi_{max} - \varphi_{min} = \pi/2$, and $r_c = \frac{1}{2}\sqrt{2}$.

can write $\sigma(A \otimes B) = \sigma(A)\,\sigma(B)$ for the eigenvalues and $\mathbf{v}_A \otimes \mathbf{v}_B$ for the eigenvectors.[21] It is then easy to verify that

$$\sigma(\mathbb{D}) = \left\{ -(1+\varepsilon)\frac{N}{2};\ (N-2)\text{-fold degenerate} \right\} \cup \{0\} \cup \{-\varepsilon N\} \quad (45)$$

As $\varepsilon \to 0$, only the last two are relevant. They belong to the eigenvectors $(1,..., 1, 1,..., 1)$ and $(1,..., 1, -1,..., -1)$. The latter direction makes the phase-locked state unstable at K_c. Its physical interpretation is quite intuitive.

6. SOME BUT NOT ALL OSCILLATORS CAN LOCK

Phase locking is apparently the rule if the ω_i do not scatter too much. A natural question then is: what happens when some but not all of the oscillators can lock? For example, for $K = 2.2815$ we take $\omega_1 = 1.5$, $\omega_2 = 2.0$, while $\omega_3 = 4.0$ and the $p(\omega_i)$ all equal 1/3. By the fixed-point equation (22) we obtain $K < K_c = 2.2816$, which slightly exceeds K. The sublattices $I(\omega_1)$ and $I(\omega_2)$ can lock, at least in principle, whereas $I(\omega_3)$ has to stay apart. If so, one might think that the frequency common to the sublattices $I(\omega_1)$ and $I(\omega_2)$ would be 1.75. This is not true, due to the exact sum rule (3). Neither do they lock exactly, nor is their "common frequency" the appropriately weighted mean of the sublattice frequencies;

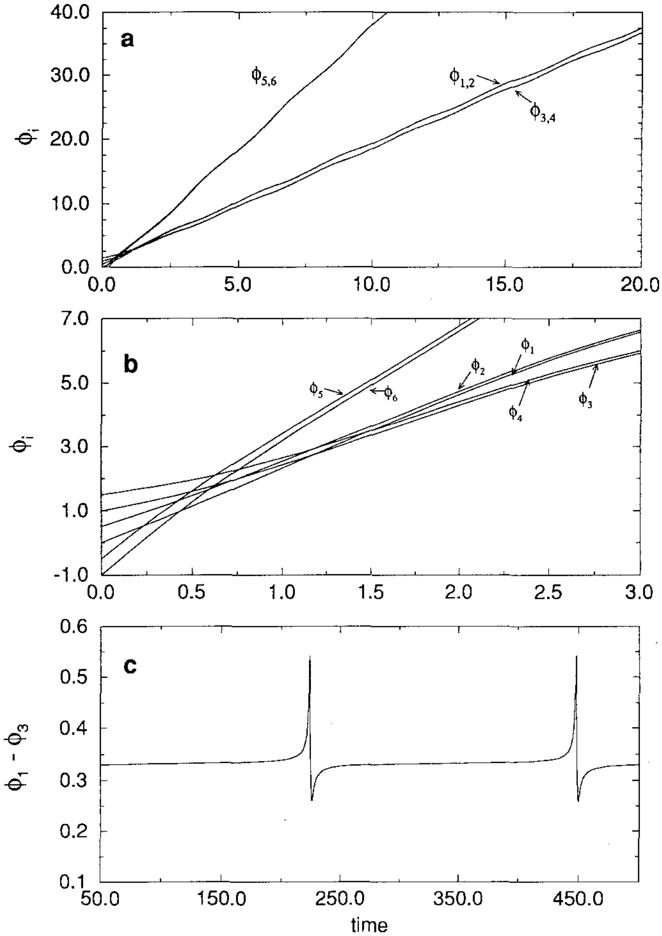

Fig. 4. (a, b) Solution of (1) for $N=6$ and $K=1$ with $\omega_1=\omega_2=2$, $\omega_3=\omega_4=1.5$, and $\omega_5=\omega_6=4$. The initial condition is the homogeneous distribution $\phi_4(0)=1.5>\phi_3(0)>\phi_2(0)>\phi_1(0)>\phi_5(0)>\phi_6(0)=-1$. Asymptotically (not shown here), ϕ_1 and ϕ_2, ϕ_3 and ϕ_4, and ϕ_5 and ϕ_6 merge pairwise, as is already suggested by (b). However, (c) exhibits $\phi_1-\phi_3$ for large times and $K=2.2815<K_c=2.2816$, and shows that the sublattices $I(1.5)$ and $I(2)$, though exhibiting something like a partial phase locking, do *not* lock exactly. Instead we find a periodic oscillation with recurrence time $T=113$, which is the distance between the two peaks, whereas for $K=1$ we would have gotten $T=4.7$.

cf. Fig. 4. Moreover, in numerical simulations it turns out that asymptotically, as $t \to \infty$, all phases $\varphi_i(t)$ on a single sublattice $I(\omega)$ approach the *same* limit $\varphi(\omega; t)$. Hence we end up with a *reduced* dynamics

$$\dot{\varphi}(\omega) = \Delta(\omega) - K \sum_{\{\omega'\}} p(\omega') \sin[\varphi(\omega) - \varphi(\omega')] \tag{46}$$

obeying the exact sum rule

$$\frac{d}{dt}\left(\sum_{\{\omega\}} p(\omega)\, \varphi(\omega)\right) = 0 \tag{47}$$

In view of the Lyapunov function (10) the reduction is easily understood. Though K is less than K_c and thus a stationary point of $\mathscr{H}$ cannot be found, the ferromagnetic interaction is at least *minimized on the sublattices*, if there the spins are parallel, i.e., $\varphi_i(t) = \varphi(\omega; t)$ for all $i \in I(\omega)$. So it is fair to call this a *sublattice phase locking*. The "minimizing path" itself depends on the distribution of the ω's. Moreover, since the "partial" phase locking of the sublattices that in principle could lock is *not* an exact one, a rigorous but simple description of the system's behavior for $K < K_c$ is hard to imagine—except for (46).

7. DISCUSSION: CRITICALITY AND UNIVERSALITY

It may be well to contrast the present results with those obtained for more "generic" models[2–13] that have an absolutely continuous frequency distribution with a symmetric and one-humped density function, such as the Gaussian and the Lorentzian, and ask whether their behavior is truly generic. In this type of model one has,[3,12] as K decreases from infinity, *two* transitions: one at K_c where the random field takes over partially in that the system is only partially phase locked, and another one at $K_{pc} < K_c$ where also the partially locked state disappears. For $K < K_{pc}$ the oscillators behave truly incoherently. Partial phase locking means that oscillators with frequencies near the center of the distribution remain locked, whereas outlying oscillators are desynchronized. In passing we note that a uniform distribution on a bounded interval has $K_{pc} = K_c$.

If partial locking were generic, a discrete frequency distribution would show a similar behavior. For example, let us return to the system of the previous section (Fig. 4) that has three frequencies, $\omega_i = 1.5$, 2, and 4. Without $\omega_i = 4$, the system would phase lock for $K > 0.5$ and we therefore expect that, if the partial-locking behavior were generic, then for $K = 2.2815 < K_c = 2.2816$ the sublattices belonging to $\omega_i = 1.5$ and 2, which are also nearest to $\langle \omega \rangle = 2.5$, would lock as well. As one sees in Fig. 4c, they do not. We have verified that they lock nowhere below K_c.

Even more can be said, however. One might object that the underlying distribution of Fig. 4 is asymmetric and that partial locking is to be expected for a symmetric distribution. To verify this we have studied the dynamics of a system with a symmetric distribution consisting of four frequencies. We have simply added a fourth $\omega = -0.5$ to the already existing three of Fig. 4. Then $\langle \omega \rangle = 1.75$ and $K_c = 3.4748$, as follows from the fixed-point equation (22). Figure 5a with $K = 3.4747 < K_c = 3.4748$ shows that, for a

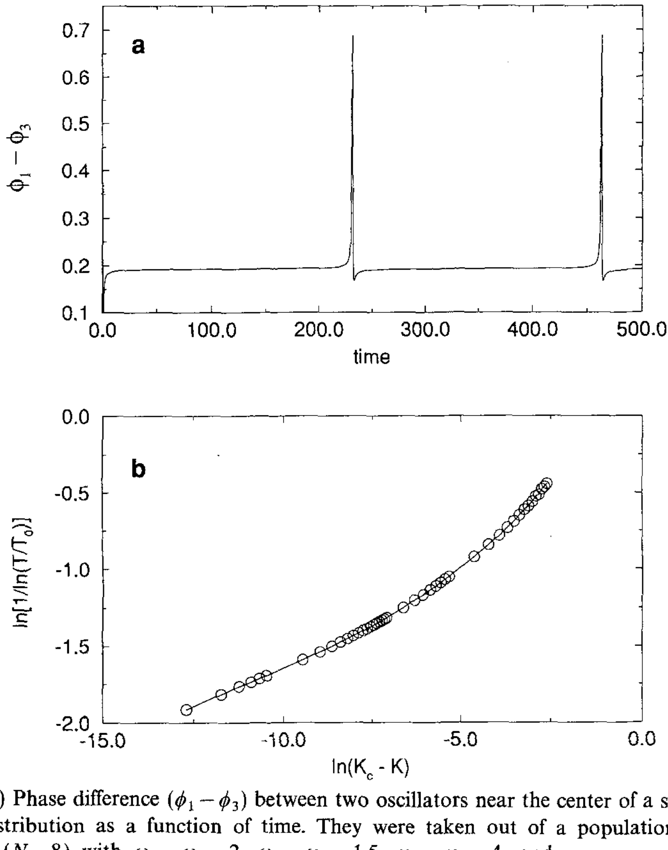

Fig. 5. (a) Phase difference $(\phi_1 - \phi_3)$ between two oscillators near the center of a symmetric, discrete distribution as a function of time. They were taken out of a population of eight oscillators $(N = 8)$ with $\omega_1 = \omega_2 = 2$, $\omega_3 = \omega_4 = 1.5$, $\omega_5 = \omega_6 = 4$, and $\omega_7 = \omega_8 = -0.5$. The initial conditions are as in Fig. 4b. Furthermore, $K = 3.4747 < K_c = 3.4748$. The recurrence time T is about 231. It is plain that the sublattices $I(1.5)$ and $I(2)$ do not lock, even though their frequencies are near the center $\langle \omega \rangle = 1.75$, K is only slightly below K_c, and, between the peaks, the system does look "partially phase locked." (b) To verify that the dependence of T upon $(K_c - K)$ has a first-order character, we have plotted $\ln[1/\ln(T/T_0)]$ against $\ln(K_c - K)$, where $K_c = 3.474828$ and $T_0 = 2.2$. The open circles represent numerically obtained data points. A pure power law behavior à la $T = T_0(K_c - K)^x$ for some $x < 0$ does not occur. The first-order dependence is in agreement with the nature of the transition as K approaches K_c from above; cf. Fig. 2 and the discussion below (22).

discrete distribution, the absence of partial locking below K_c is quite universal. Here, too, we have verified that the two sublattices $I(1.5)$ and $I(2)$ associated with frequencies near the center of the distribution lock nowhere below K_c. One may wonder, though, how the system "feels" that K is approaching K_c from below. To this end we have studied the recurrence time T of the asymptotic phase difference between $I(1.5)$ and $I(2)$ as a function of $(K_c - K)$; cf. Fig. 5b. As K approaches K_c from below, the amplitude of the phase difference does not vary, but the recurrence time T does: it diverges to infinity. As in a *first-order* phase transition, we do not find a pure power law behavior. For $K > K_c$ the phase difference is asymptotically fixed, i.e., $T = \infty$.

There are apparently classes of oscillator models with different "generic" behavior. That is, there are different *universality classes.* The absolutely continuous distributions[2–13] belong to one class and the discrete distributions to another one. The former give rise to two transitions at K_c and K_{pc}, whereas the latter appear to have only a single transition at K_c. We now want to discuss the character of the ground states in more detail.

It is the very existence of a Lyapunov function $\mathscr{H}$ that allows a physically transparent treatment of a phase-locked state of the Kuramoto model as a ground state. In fact, the argument holds for any equivalent model with finite-range interactions.[9,22–24] We first list some interesting data concerning asymptotic states and their dependence upon the dimension in models with finite-range interactions. For example, Sakaguchi *et al.*[23,24] have studied the Gaussian case and numerically obtained the result that *no* extensive synchronized clusters exist for dimensions $d \leqslant 2$, whereas they state that for $2 < d \leqslant 4$ the order parameter r vanishes, even though they cannot exclude that extensive synchronized clusters do exist. In their own words,[24] the phase difference becomes indefinitely large with distance if $d \leqslant 4$. The underlying physics of these and similar results is a characterization of the ground states of the underlying spin model.

The Lyapunov function $\mathscr{H}$ has two constituents, an XY ferromagnet with coupling strength $K > 0$ and a random field $(\omega_i - \langle\omega\rangle)$; cf. (10). In the case of the Kuramoto model, the XY ferromagnet dominates for $K > K_c$, the sublattices have a homogeneous phase, and their phases lock with respect to each other. The larger is $|\omega - \langle\omega\rangle|$, the larger is the phase shift. For $K < K_c$, the sublattices still have a homogeneous phase as $t \to \infty$ (sublattice phase locking), but $\mathscr{H}$ has no stationary point and the "minimizing path" is determined by the distribution of the ω's, as is brought out by (46).

For a d-dimensional oscillator model with ferromagnetic nearest-neighbor interactions, the interpretation of the numerical work[9,23,24] is more complicated. Due to results of Aizenman and Wehr,[25] we expect *no* long-range order in an XY ferromagnet with random field for dimensions $d \leqslant 4$. Its exact ground states are not known yet, but it may well be that their character for $d \leqslant 2$ is different from that for $2 < d \leqslant 4$ in that for $d \leqslant 2$, ground states more or less follow the random field, whereas in higher dimensions, large "magnetized" clusters or blobs, each associated with a specific frequency, occur—as is also suggested by the numerics.[9,23,24] A large-deviation argument of Strogatz and Mirollo[9] shows that, if these clusters are to be bulky, i.e., nonfractal, then they all have to be *finite*—except for the one which is associated with $\langle \omega \rangle$. That is, as these authors say, extensive clusters of synchronized oscillators would have to be spongelike, except for one.

Though interactions more general than the ferromagnetic one and the addition of white noise can be treated in a similar vein, we have refrained from doing so here. A short comment on oscillator models for neural networks may be in order, though.

The model (1) with Hebbian couplings à la

$$J_{ij} = N^{-1} \sum_{\mu} \xi_i^{\mu} \xi_j^{\mu} \tag{48}$$

for $1 \leqslant \mu \leqslant q$ random patterns (ξ_i^{μ}; $1 \leqslant i \leqslant N$) has been used[26] to elegantly describe coherent oscillations which have been found in the visual cortex.[27,28] The dynamics of a neuron is taken into account by a single phase ϕ_i and we can say the neuron fires if, for instance, $\phi_i \approx 0$. In a network[26] neurons are to communicate via $J_{ij} \sin(\phi_i - \phi_j)$. One now may ask: How good is such a description? More precisely, is there any cognitive meaning in classifying these "neurons" according to the sublattice $I(\omega)$ they belong to, and is a collection of *spiking* neurons really equivalent (in some sense) to a set of nonlinear oscillators à la (1)?

According to Hebb,[29] the synapses, not the neurons themselves, store the information. This is brought out by (48). The neurons "only" interpret the signals mediated by the synapses. By itself, a neuron is a *threshold* device and its activity is an all-or-none process, whereas the oscillators à la (1) are just the opposite. They feel each other all the time, whereas real neurons notice each other *only if and when they spike*. So, in contrast to the above phase description, both the neuron's input and its output are all-or-none phenomena. Though here, too, we can obtain phase locking, the consequences for data processing in a system of spiking neurons are of a different nature. For instance,[30,31] pattern segmentation works much better. In summary, the Kuramoto model offers a vast playing ground to test locking phenomena, but its relevance to practical problems should not be overestimated.

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